

LARGE CONJUGACY CLASSES, PROJECTIVE FRAÏSSÉ LIMITS, AND THE PSEUDO-ARC

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ABSTRACT. We show that the automorphism group, $\text{Aut}(\mathbb{P})$, of a projective Fraïssé limit \mathbb{P} , whose natural quotient is the pseudo-arc, has a comeager conjugacy class. This generalizes an unpublished result of Oppenheim that $\text{Aut}(\mathbb{P})$ (and consequently, the group of all homeomorphisms of the pseudo-arc) has a dense conjugacy class. We also present a simple proof of the result of Oppenheim.

1. INTRODUCTION

1.1. The pseudo-arc. The *pseudo-arc* P is the unique hereditary indecomposable chainable continuum. Recall that a continuum is a compact and connected metric space; it is indecomposable if it is not a union of two proper subcontinua, and it is hereditary indecomposable if every subcontinuum is indecomposable. We call a continuum chainable if each open cover of it is refined by an open cover U_1, U_2, \dots, U_n such that for i, j , $U_i \cap U_j \neq \emptyset$ if and only if $|j - i| \leq 1$.

The pseudo-arc has a remarkably rich structure, for example, it is injectively ultrahomogeneous (see [1] and [5]). Irwin and Solecki [3] discovered that it is also projectively ultrahomogeneous. Moreover, the collection of all subcontinua of $[0, 1]^{\mathbb{N}}$ homeomorphic to the pseudo-arc is comeager in the space of all subcontinua of $[0, 1]^{\mathbb{N}}$, equipped with the Hausdorff metric. For more information on the pseudo-arc, see [6].

1.2. Projective Fraïssé theory. We recall here basic notions and results on the projective Fraïssé theory, developed by Irwin and Solecki in [3].

Given a language L that consists of relation symbols $\{r_i\}_{i \in I}$, and function symbols $\{f_j\}_{j \in J}$, a *topological L-structure* is a compact zero-dimensional second-countable space A equipped with closed relations r_i^A and continuous functions f_j^A , $i \in I, j \in J$. A continuous surjection $\phi: B \rightarrow A$ is an *epimorphism* if it preserves the structure, more precisely, for a function symbol f of arity n and $x_1, \dots, x_n \in B$ we require:

$$f^A(\phi(x_1), \dots, \phi(x_n)) = \phi(f^B(x_1, \dots, x_n));$$

and for a relation symbol r of arity m and $x_1, \dots, x_m \in B$ we require:

$$\begin{aligned} &r^A(x_1, \dots, x_m) \\ \iff &\exists y_1, \dots, y_m \in B (\phi(y_1) = x_1, \dots, \phi(y_m) = x_m, \text{ and } r^B(y_1, \dots, y_m)). \end{aligned}$$

By an *isomorphism* we mean a bijective epimorphism.

For the rest of this section fix a language L . Let \mathcal{G} be a family of finite topological L -structures. We say that \mathcal{G} is a *projective Fraïssé family* if the following two conditions hold:

(F1) (the joint projection property: JPP) for any $A, B \in \mathcal{F}$ there are $C \in \mathcal{F}$ and epimorphisms from C onto A and from C onto B ;

(F2) (the amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and any epimorphisms $\phi_1: B_1 \rightarrow A$ and $\phi_2: B_2 \rightarrow A$, there exist $C \in \mathcal{F}$, $\phi_3: C \rightarrow B_1$, and $\phi_4: C \rightarrow B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

A topological L -structure \mathbb{P} is a *projective Fraïssé limit* of \mathcal{G} if the following three conditions hold:

(L1) (the projective universality) for any $A \in \mathcal{F}$ there is an epimorphism from \mathbb{P} onto A ;

(L2) for any finite discrete topological space X and any continuous function $f: \mathbb{P} \rightarrow X$ there are $A \in \mathcal{F}$, an epimorphism $\phi: \mathbb{P} \rightarrow A$, and a function $f_0: A \rightarrow X$ such that $f = f_0 \circ \phi$.

(L3) (the projective ultrahomogeneity) for any $A \in \mathcal{F}$ and any epimorphisms $\phi_1: \mathbb{P} \rightarrow A$ and $\phi_2: \mathbb{P} \rightarrow A$ there exists an isomorphism $\psi: \mathbb{P} \rightarrow \mathbb{P}$ such that $\phi_2 = \phi_1 \circ \psi$;

Here is the fundamental result in the projective Fraïssé theory:

Theorem 1.1 (Irwin-Solecki, [3]). *Let \mathcal{F} be a countable projective Fraïssé family of finite topological L -structures. Then:*

(1) *there exists a projective Fraïssé limit of \mathcal{F} ;*

(2) *any two topological L -structures that are projective Fraïssé limits are isomorphic.*

In the proposition below we state some properties of the projective Fraïssé limit.

Proposition 1.2. (1) If \mathbb{P} is the projective Fraïssé limit the following condition (called *the extension property*) holds: Given $\phi_1: B \rightarrow A$, $A, B \in \mathcal{F}$, and $\phi_2: \mathbb{P} \rightarrow A$, then, there is $\psi: \mathbb{P} \rightarrow B$ such that $\phi_2 = \phi_1 \circ \psi$.

(2) If \mathbb{P} satisfies the projective universality (L1), the extension property, and (L2), then it also satisfies projective ultrahomogeneity, and therefore is isomorphic to the projective Fraïssé limit.

1.3. The pseudo-arc as a projective Fraïssé limit. Let $H(P)$ denote the group of all homeomorphisms of the pseudo-arc. Let L_0 be the language that consists of one binary relation symbol r . Let \mathcal{G} denote the family of finite reflexive linear graphs, more precisely, we say that $A = ([n], r^A)$, where $[n] = \{1, 2, \dots, n\}$ is a finite reflexive linear graph if $r^A(x, y)$ holds if and only if $x = y$, or $x = i, y = i + 1$ for some $i = 1, 2, \dots, n - 1$, or $x = i + 1, y = i$ for some $i = 1, 2, \dots, n - 1$.

Recall the following results obtained by Irwin and Solecki [3].

Theorem 1.3 (Theorem 3.1 in [3]). *The family \mathcal{G} is a projective Fraïssé family.*

Lemma 1.4 (Lemma 4.1 in [3]). *Let \mathbb{P} be the projective Fraïssé limit of \mathcal{G} . Then $r^{\mathbb{P}}$ is an equivalence relation whose each equivalence class has at most two elements.*

Theorem 1.5 (Theorem 4.2 in [3]). *Let \mathbb{P} be the projective Fraïssé limit of \mathcal{G} . Then $\mathbb{P}/r^{\mathbb{P}}$ is the pseudo-arc.*

1.4. Results. The existence of a comeager conjugacy class was verified for various important non-archimedean groups, that is Polish (separable and completely metrizable topological) groups that have a neighborhood basis of the identity that consists of open subgroups. This class of groups coincides with the class of automorphism groups of countable model-theoretic structures. A few examples of groups with a comeager conjugacy class are: the automorphism group of rationals, the automorphism group of the random graph, the automorphism group of the rational Urysohn space, the homeomorphism group of the Cantor set (all of these groups, except the automorphism group of the rationals, enjoy an even a stronger property, called ample generics, for the definition see [4]). For more on this topic, see [4] and [2].

The main result of the present paper is the following theorem.

Theorem 1.6. *The group of all automorphisms of \mathbb{P} , $\text{Aut}(\mathbb{P})$, has a comeager conjugacy class.*

The proof of Theorem 1.6 will be given in Section 3. This result strengthens the result of Oppenheim that $\text{Aut}(\mathbb{P})$ has a dense conjugacy class. We give a simple and self-contained proof of his result in Section 2. In the same section, using results from [3], we show how the existence of a dense conjugacy class in $\text{Aut}(\mathbb{P})$ implies the existence of a dense conjugacy class in the group of all homeomorphisms of the pseudo-arc, $H(P)$. In Appendix A, we present criteria for an automorphism group of a projective Fraïssé limit to have, respectively, a dense conjugacy class and a comeager conjugacy class.

2. DENSE CONJUGACY CLASS IN $\text{Aut}(\mathbb{P})$ AND $H(P)$

In this section we present a short and simple proof of an unpublished result of Oppenheim that the group of all automorphisms of \mathbb{P} has a dense conjugacy class. It follows from [3] and is shown below that this result easily implies that the group of all homeomorphisms of the pseudo-arc has a dense conjugacy class. Both Oppenheim's proof and the proof presented here use the projective Fraïssé theory. Oppenheim shows a version of Proposition 2.5 for a different family than the family \mathcal{F} investigated by us.

Theorem 2.1 (Oppenheim). *The group of all homeomorphisms of the pseudo-arc, $H(P)$, has a dense conjugacy class.*

A function $f: \mathbb{P} \rightarrow \mathbb{P}$ is an automorphism if and only if it is a homeomorphism and for every $x, y \in \mathbb{P}$, $r^{\mathbb{P}}(x, y) \iff r^{\mathbb{P}}(f(x), f(y))$.

Theorem 2.2 (Oppenheim). *The group of all automorphisms of \mathbb{P} , $\text{Aut}(\mathbb{P})$, has a dense conjugacy class.*

We first see how Theorem 2.2 implies Theorem 2.1.

Proof of Theorem 2.1. The space $\text{Aut}(\mathbb{P})$ can be identified with a dense subspace of $H(P)$. This follows from Lemma 4.8 in the Irwin-Solecki paper [3] (take $X = P$, an arbitrary $f_1 \in H(P)$, and take $f_2 = \text{id}$). \square

Let $L = L_0 \cup \{s\} = \{r, s\}$, where s is a symbol for a binary relations. With some abuse of notation, we will be writing (A, s^A) , where $A = (A, r^A)$ is a topological L_0 -structure, instead of (A, r^A, s^A) , whenever (A, r^A, s^A) is a topological L -structure.

Let $f \in \text{Aut}(\mathbb{P})$. We view (\mathbb{P}, f) as a topological L -structure $(\mathbb{P}, s^{\mathbb{P}})$, where $s^{\mathbb{P}}(x, y) \iff (x, y) \in \text{graph}(f) \iff f(x) = y$. Define

$$\begin{aligned} \mathcal{F} = & \{(A, s^A) : A \in \mathcal{G} \text{ and } \exists \phi: \mathbb{P} \rightarrow A \exists f \in \text{Aut}(\mathbb{P}) \text{ such that} \\ & \phi: (\mathbb{P}, f) \rightarrow (A, s^A) \text{ is an epimorphism}\}. \end{aligned}$$

Remark 2.3. Let $f \in \text{Aut}(\mathbb{P})$. Let $A \in \mathcal{G}$. For a given epimorphism $\phi: \mathbb{P} \rightarrow A$ we can talk about a restriction of f to A :

$$f \upharpoonright A = \{(a, b) \in A^2 : \phi^{-1}(a) \cap f(\phi^{-1}(b)) \neq \emptyset\}.$$

It is not difficult to see that

$$\mathcal{F} = \{(A, s^A) : A \in \mathcal{G} \text{ and } \exists \phi: \mathbb{P} \rightarrow A \exists f \in \text{Aut}(\mathbb{P}) f \upharpoonright A = s^A\}.$$

Lemma 2.4. *Let $(A, s^A) \in \mathcal{F}$. Let $\phi: \mathbb{P} \rightarrow A$ be such that $\phi: (\mathbb{P}, f) \rightarrow (A, s^A)$ is an epimorphism. Let $\psi: \mathbb{P} \rightarrow A$ be an epimorphism. Then there is $g \in \text{Aut}(\mathbb{P})$ such that $\psi: (\mathbb{P}, gfg^{-1}) \rightarrow (A, s^A)$ is an epimorphism.*

Proof. Using the projective universality, get $g \in \text{Aut}(\mathbb{P})$ such that $\psi \circ g = \phi$. This g works. \square

We will use several times Lemma 2.4 in proofs of Propositions 2.6 and 3.3 without mentioning it.

Recall from the Introduction that \mathcal{F} has the JPP if and only if for every $(A, s^A), (B, s^B) \in \mathcal{F}$ there is $(C, s^C) \in \mathcal{F}$ and epimorphisms from (C, s^C) onto (A, s^A) and from (C, s^C) onto (B, s^B) .

We split the proof of Theorem 2.2 into two propositions.

Proposition 2.5. The family \mathcal{F} has the JPP.

Proposition 2.6. The property JPP for \mathcal{F} implies $\text{Aut}(\mathbb{P})$ has a dense conjugacy class.

The proof of Proposition 2.6 will be an adaptation to our context of the proof of one of the directions of Theorem 2.1 in [4].

For $(A, s^A) \in \mathcal{F}$ and an epimorphism $\phi: \mathbb{P} \rightarrow A$ define

$$[\phi, s^A] = \{f \in \text{Aut}(\mathbb{P}): \phi: (\mathbb{P}, f) \rightarrow (A, s^A) \text{ is an epimorphism}\}.$$

Sets of the form $[\phi, s^A]$ are clopen in $\text{Aut}(\mathbb{P})$, where the topology on $\text{Aut}(\mathbb{P})$ is induced from the uniform convergence topology on $H(2^\mathbb{N})$, the group of all homeomorphisms of the Cantor set $2^\mathbb{N}$ (recall that the underlying set of \mathbb{P} is equal to $2^\mathbb{N}$).

Lemma 2.7. *The family of all sets $[\phi, s^A]$, where $(A, s^A) \in \mathcal{F}$, is a basis of the topology on $\text{Aut}(\mathbb{P})$.*

Proof. Take $g \in \text{Aut}(\mathbb{P})$, $\epsilon > 0$, and $U = \{f \in \text{Aut}(\mathbb{P}): \forall x \ d(f(x), g(x)) < \epsilon\}$ (d is any metric on the underlying set of \mathbb{P}). This is an open set. We want to find a clopen neighborhood of g that is of the form $[\phi, s^A]$ and is contained in U . For this, take an arbitrary partition Q of \mathbb{P} of mesh $< \epsilon$ and let $P = \{q_0 \cap g^{-1}(q_1): q_0, q_1 \in Q\}$. Let A be a refinement of P such that A together with the relation r^A inherited from $r^\mathbb{P}$ is in \mathcal{G} (condition (L2) guarantees the existence of such A). Let ϕ be the natural projection from \mathbb{P} to A . By the choice of A , this is an epimorphism. We let $s^A = \{(p, r) \in A^2: \exists_{x \in p} \exists_{y \in r} g(x) = y\}$. Clearly $g \in [\phi, s^A]$ and $(A, s^A) \in \mathcal{F}$. Take any $p \in A$, say $p \subseteq q_0 \cap g^{-1}(q_1)$, $q_0, q_1 \in P$. Then $g(p) \subseteq q_1$. Now take any $f \in [\phi, s^A]$ and notice that $f(p) \subseteq q_1$. Since $\text{diam}(q_1) < \epsilon$ and $p \in A$ was arbitrary, we get $f \in U$. \square

For $(A, s^A) \in \mathcal{F}$ and an epimorphism $\phi: \mathbb{P} \rightarrow A$ define

$$D(\phi, s^A) = \{f \in \text{Aut}(\mathbb{P}): \exists g \in \text{Aut}(\mathbb{P}) gfg^{-1} \in [\phi, s^A]\}.$$

This set is open.

Lemma 2.8. *The set $D(\phi, s^A)$, where $(A, s^A) \in \mathcal{F}$, is dense.*

Proof. Fix $D(\phi, s^A)$ and take $[\psi, s^B]$. We show that $D(\phi, s^A) \cap [\psi, s^B] \neq \emptyset$. Since sets $[\psi, s^B]$ form a basis, this will finish the proof. Using the JPP, take $(C, s^C) \in \mathcal{F}$ and epimorphisms $\alpha: (C, s^C) \rightarrow (A, s^A)$ and $\beta: (C, s^C) \rightarrow (B, s^B)$. Using the extension property, find $\gamma: \mathbb{P} \rightarrow C$ and $\delta: \mathbb{P} \rightarrow C$ such that $\phi = \alpha \circ \gamma$ and $\psi = \beta \circ \delta$. Take any $f \in [\delta, s^C] \subseteq [\psi, s^B]$ and take $g \in \text{Aut}(\mathbb{P})$ such that $gfg^{-1} \in [\gamma, s^C]$. Then $gfg^{-1} \in [\phi, s^A]$, and therefore $D(\phi, s^A) \cap [\psi, s^B] \neq \emptyset$. \square

Proof of Proposition 2.6. The intersection of all $D(\phi, s^A)$ is open and dense, in particular it is nonempty. From the definition of $D(\phi, s^A)$, every function in this intersection has a dense conjugacy class. \square

To prove Proposition 2.5 we need Lemmas 2.12 and 2.13.

Definition 2.9. Let $A \in \mathcal{G}$ or $A = \mathbb{P}$. Let s^A be a binary relation on A .

(1) We say that s^A is *surjective* if for every $a \in A$ there are $b, c \in A$ such that $s^A(a, b)$ and $s^A(c, a)$.

(2) We say that s^A is *connected* if the graph $G_{(A, s^A)}$ that has $s^A = \{(a, b) : s^A(a, b)\}$ as the set of vertices and $\{((a, b), (c, d)) : r^A(a, c), r^A(b, d)\}$ as the set of edges, satisfies the following: for any clopen $X \subseteq s^A$ such that X and $A \setminus X$ are nonempty there are $x \in X$ and $y \in s^A \setminus X$ such that x and y are joined by an edge. (The topology on s^A is inherited from the product topology on $A \times A$. If $A \in \mathcal{G}$, $A \times A$ is discrete.)

(3) We say that s^A is an *antidiagonal* of $A = [n] = \{1, 2, \dots, n\}$ if $s^A = \{(k, n+1-k) : k = 1, 2, \dots, n\}$.

Remark 2.10. Note that for a finite $A \in \mathcal{G}$, s^A is connected if and only if $G_{(A, s^A)}$ is connected as a graph (that is, every two vertices are connected by a path).

Example 2.11. (1) Take $A = (\{1, 2, 3, 4\}, s^A)$, where $s^A = \{(1, 3), (2, 3), (3, 1), (3, 2), (3, 4), (4, 1)\}$. Then s^A is connected.

(2) Take $A = (\{1, 2, 3, 4\}, s^A)$, where $s^A = \{(1, 2), (2, 1), (2, 4), (3, 3), (3, 4), (4, 2)\}$. Then s^A is not connected.

The next lemma is due to Solecki.

Lemma 2.12. For any (A, s^A) with $A \in \mathcal{G}$ and s^A surjective and connected there is (B, s^B) such that: s^B is the antidiagonal of B and there exists an epimorphism from (B, s^B) onto (A, s^A) .

Proof of Lemma 2.12. Take (A, s^A) with $A \in \mathcal{G}$ and s^A surjective and connected. Write $A = [k]$. We let $[k] \times [k]$ to be the product graph. Since s^A surjective and connected, there is i_0 such that $(i_0, i_0) \in s^A$ or $(i_0, i_0+1) \in s^A$ or $(i_0+1, i_0) \in s^A$. Let $h: [m] \rightarrow s^A$, for some m , be a surjective graph homomorphism with $h(1) = (i_0, i_0)$ or $h(1) = (i_0, i_0+1)$ or $h(1) = (i_0+1, i_0)$, according to the cases above. Define $\phi: [4m] \rightarrow [k]$ as follows:

$$\phi(i) = \begin{cases} \pi_1(h(i)) & \text{if } 1 \leq i \leq m, \\ \pi_1(h(2m - i + 1)) & \text{if } m + 1 \leq i \leq 2m, \\ \pi_2(h(i - 2m)) & \text{if } 2m + 1 \leq i \leq 3m, \\ \pi_2(h(4m - i + 1)) & \text{if } 3m + 1 \leq i \leq 4m. \end{cases}$$

Let $s^B \subseteq [4m] \times [4m]$ be the antidiagonal of $B = [4m]$. Then, just by applying the above formulas, we see that $\phi: (B, s^B) \rightarrow (A, s^A)$ is an epimorphism. □

Lemma 2.13. Let $A \in \mathcal{G}$. Then $(A, s^A) \in \mathcal{F}$ if and only if s^A is surjective and connected.

Proof. Suppose that $(A, s^A) \in \mathcal{F}$. Let $\phi: (\mathbb{P}, f) \rightarrow (A, s^A)$ be an epimorphism. Since f is a homeomorphism, $\text{graph}(f) = \{(x, f(x)) : x \in \mathbb{P}\}$ is a surjective relation on \mathbb{P} .

We show that $G_{(\mathbb{P}, \text{graph}(f))}$ is connected. Suppose towards a contradiction that there is a clopen set $X \subseteq \text{graph}(f)$ such that X and $\text{graph}(f) \setminus X$ are nonempty, and there are no $(x, f(x)) \in X$, $(y, f(y)) \in \text{graph}(f) \setminus X$ such that $r^{\mathbb{P}}(x, y)$ (and $r^{\mathbb{P}}(f(x), f(y))$). Let Y be the projection of X into the first coordinate. Then Y and $\mathbb{P} \setminus Y$ are nonempty clopen and for no $x \in Y$ and $y \in \mathbb{P} \setminus Y$, $r^{\mathbb{P}}(x, y)$. However, this is impossible (apply (L2) to $A = \{Y, \mathbb{P} \setminus Y\}$ and the natural projection from \mathbb{P} to A). Finally, observe that surjective relations and connected relations are preserved by epimorphisms.

For the other direction, take (A, s^A) such that $A \in \mathcal{G}$ and s^A is surjective and connected. Take (B, s^B) and $\phi: (B, s^B) \rightarrow (A, s^A)$ such that s^B is the antidiagonal of B and ϕ is an epimorphism. Using the projective universality, take any epimorphism $\psi_1: \mathbb{P} \rightarrow B$. Let $\text{inv}: B \rightarrow B$, the ‘inverse’ of B , be the only nontrivial automorphism of B (provided that B has at least two elements, which we can assume). Let $\psi_2 = \text{inv} \circ \psi_1$. From the projective ultrahomogeneity applied to ψ_1 and ψ_2 , we get $f \in \text{Aut}(\mathbb{P})$ such that $\psi_1 \circ f = \psi_2$. Then $\phi \circ \psi_1: (\mathbb{P}, f) \rightarrow (A, s^A)$ is the required epimorphism. \square

Proof of Proposition 2.5. Take $(A, s^A), (B, s^B) \in \mathcal{F}$. Then s^A and s^B are surjective and connected. Without loss of generality, s^A and s^B are antidiagonals of A and B , respectively. Write $A = [k]$ and $B = [n]$. Take $C = [kn]$ and let s^C be the antidiagonal of C . We show that this (C, s^C) works. For this, take $\phi_1: (C, s^C) \rightarrow (A, s^A)$ given by $\phi_1((i-1)n+j) = i$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, n$, and take $\phi_2: (C, s^C) \rightarrow (B, s^B)$ given by $\phi_2((i-1)n+j) = i$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, k$. \square

3. COMEAGER CONJUGACY CLASS IN $\text{Aut}(\mathbb{P})$

In this section we show our main theorem.

Theorem 3.1. *The group of all automorphisms of \mathbb{P} , $\text{Aut}(\mathbb{P})$, has a comeager conjugacy class.*

We say that \mathcal{F} has the *coinitial amalgamation property* (the CAP) if and only if for every $(A_0, s^{A_0}) \in \mathcal{F}$ there is $(A, s^A) \in \mathcal{F}$ and an epimorphisms $\psi: (A, s^A) \rightarrow (A_0, s^{A_0})$ such that for every $(B, s^B), (C, s^C) \in \mathcal{F}$ and epimorphism $\phi_1: (B, s^B) \rightarrow (A, s^A)$ and $\phi_2: (C, s^C) \rightarrow (A, s^A)$ there is $(D, s^D) \in \mathcal{F}$ and epimorphisms $\phi_3: (D, s^D) \rightarrow (B, s^B)$ and $\phi_4: (D, s^D) \rightarrow (C, s^C)$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

We split the proof of Theorem 3.1 into Propositions 3.2 and 3.3.

Proposition 3.2. The family \mathcal{F} has the CAP. More precisely, the coinitial family \mathcal{D} , defined in Lemma 3.4 below, has the AP.

Since we already know that \mathcal{F} has the JPP, Proposition 3.2 together with Proposition 3.3 will finish the proof of Theorem 3.1.

Proposition 3.3. Properties CAP and JPP for \mathcal{F} imply $\text{Aut}(\mathbb{P})$ has a comeager conjugacy class.

It will be convenient for us to work only with those structures that have an even number of elements.

Lemma 3.4. *The family*

$$\mathcal{D} = \{(A, s^A) \in \mathcal{F}: s^A \text{ is the antidiagonal of } A \text{ and } |A| \text{ is an even number}\}$$

is coinitial in \mathcal{F} .

Proof. We know already that $\mathcal{D}_0 = \{(A, s^A) \in \mathcal{F}: s^A \text{ is the antidiagonal of } A\}$ is coinitial in \mathcal{F} (Lemma 2.12). Take $(A, s^A) \in \mathcal{D}_0$. Write $A = [k]$. Let $B = [2k]$ and let s^B be the antidiagonal of B . Take $\phi: (B, s^B) \rightarrow (A, s^A)$ given by $\phi((i-1)n+j) = i$, $i = 1, 2, \dots, k$, $j = 1, 2$. This is an epimorphism. \square

The proof of Proposition 3.3 will be an adaptation to our context of the proof of one of the directions of Theorem 3.4 in [4]. In the proof we use the following proposition.

Proposition 3.5 (Proposition 3.2 in [4]). Let G be a non-archimedean group. Let $f \in G$. Then the following are equivalent:

- (1) the orbit $\{gfg^{-1}: g \in G\}$ is non-meager;
- (2) for each open subgroup $V < G$, $\{gfg^{-1}: g \in V\}$ is somewhere dense;
- (3) for each open subgroup $V < G$, $x \in \text{Int}(\overline{\{gfg^{-1}: g \in V\}})$.

For $(A, s^A) \in \mathcal{F}$ and an epimorphism $\phi: \mathbb{P} \rightarrow A$ we say that $((B, s^B), \psi, \bar{\psi})$ is an extension of $((A, s^A), \phi)$ if $(B, s^B) \in \mathcal{F}$, $\psi: \mathbb{P} \rightarrow B$ is an epimorphism, $\bar{\psi}: (B, s^B) \rightarrow (A, s^A)$ is an epimorphism, and $\phi = \bar{\psi} \circ \psi$.

Proof of Proposition 3.3. We show that there is $f \in \text{Aut}(\mathbb{P})$ with a dense and non-meager orbit. Clearly such f has a comeager orbit. For $(A, s^A) \in \mathcal{F}$ and an epimorphism $\phi: \mathbb{P} \rightarrow A$ we defined in Section 2

$$[\phi, s^A] = \{f \in \text{Aut}(\mathbb{P}): \phi: (\mathbb{P}, f) \rightarrow (A, s^A) \text{ is an epimorphism}\},$$

$$D(\phi, s^A) = \{f \in \text{Aut}(\mathbb{P}): \exists g \in \text{Aut}(\mathbb{P}) gfg^{-1} \in [\phi, s^A]\},$$

and we showed that every $D(\phi, s^A)$ is open and dense.

We need some more definitions. Let $(A, s^A) \in \mathcal{F}$ and $\phi: \mathbb{P} \rightarrow A$ be an epimorphism. Let id_A be the surjective relation on A satisfying $\text{id}_A(x, y) \iff x = y$. Let

$$c(\phi, f) = \{gfg^{-1}: g \in [\phi, \text{id}_A]\}.$$

Let $((A_m, s^{A_m}), \phi_m, \bar{\phi}_m)$ list all extensions of $((A, s^A), \phi)$ such that additionally $(A_m, s^{A_m}) \in \mathcal{D}$. Further, for a given m , let $((A_m^n, s^{A_m^n}), \phi_m^n, \bar{\phi}_m^n)$ list all extensions of $((A_m, s^{A_m}), \phi_m)$. Define

$$E(\phi, s^A) = \{f \in \text{Aut}(\mathbb{P}): \text{ if } f \in [\phi, s^A] \text{ then for some } m, f \in [\phi_m, s^{A_m}]\}$$

and let

$$F_{m,n}(\phi, s^A) = \{f \in \text{Aut}(\mathbb{P}): \text{ if } f \in [\phi_m, s^{A_m}] \text{ then } c(\phi, f) \cap [\phi_m^n, s^{A_m^n}] \neq \emptyset\}.$$

Let $(A, s^A) \in \mathcal{F}$ and $\phi: \mathbb{P} \rightarrow A$ be an epimorphism. Observe that $E(\phi, s^A)$ and $F_{m,n}(\phi, s^A)$ are open and that $E(\phi, s^A)$ is dense. Claims 1, 2, and 3 will finish the proof of the proposition.

Claim 1. Let $(A, s^A) \in \mathcal{F}$. The collection of all extensions of $((A, s^A), \phi)$ form a basis of $[\phi, s^A]$.

Proof. The proof of this claim goes along the lines of the proof of Lemma 2.7. \square

Claim 2. The set $F_{m,n}(\phi, s^A)$ is dense.

Proof. Take $(B, s^B) \in \mathcal{F}$ and an epimorphism $\psi: \mathbb{P} \rightarrow B$. We show that $F_{m,n}(\phi, s^A) \cap [\psi, s^B] \neq \emptyset$. We can assume that for some ψ and $\bar{\psi}$, $((B, s^B), \psi, \bar{\psi})$ is an extension of $((A_m, s^{A_m}), \phi_m)$. Since $(A_m, s^{A_m}) \in \mathcal{D}$, there is $(C, s^C) \in \mathcal{F}$ and there are $\alpha: (C, s^C) \rightarrow (A_m^n, s^{A_m^n})$ and $\beta: (C, s^C) \rightarrow (B, s^B)$ such that $\phi_m^n \circ \alpha = \bar{\psi} \circ \beta$. Using the extension property for \mathcal{G} , take $\gamma: \mathbb{P} \rightarrow C$ and $\delta: \mathbb{P} \rightarrow C$ such that $\alpha \circ \gamma = \phi_m^n$ and $\beta \circ \delta = \psi$. Take any $f \in [s^C, \delta] \subseteq [\psi, s^B] \subseteq [\phi_m, s^{A_m}]$. We want to show $f \in F_{m,n}(\phi, s^A)$. Take $g \in \text{Aut}(\mathbb{P})$ such that $\gamma \circ g = \delta$. Then $gfg^{-1} \in [\gamma, s^C] \subseteq [\phi_m^n, s^{A_m^n}]$. Since $\phi_m^n \circ \alpha \circ \gamma = \bar{\psi} \circ \beta \circ \delta = \phi_m$, we have $g \in [\phi_m, \text{id}_{A_m}] \subseteq [\phi, \text{id}_A]$, so $gfg^{-1} \in c(\phi, f)$, and we are done. \square

Claim 3. Whenever f is in the intersection of all $D(\phi, s^A)$, $E(\phi, s^A)$, and $F_{m,n}(\phi, s^A)$, where $(A, s^A) \in \mathcal{F}$ and $\phi: \mathbb{P} \rightarrow A$ is an epimorphism, then it has a comeager conjugacy class.

Proof. We already know that such f has a dense conjugacy class. We show that the conjugacy class of f is also non-meager. Since $\{[\phi, \text{id}_A]: A \in \mathcal{G}, \phi: \mathbb{P} \rightarrow A \text{ is an epimorphism}\}$ form a basis of the identity that consists of open subgroups, via Proposition 3.5, it suffices to show that for a given $A \in \mathcal{G}$ and an epimorphism $\phi: \mathbb{P} \rightarrow A$, $c(\phi, f)$ is somewhere dense.

Take s^A satisfying $s^A(a, b)$ if and only if there are $x, y \in \mathbb{P}$ such that $\phi(x) = a$, $\phi(y) = b$, and $f(x) = y$. Then $f \in [\phi, s^A]$. Since $f \in E(\phi, s^A)$, for some m , $f \in [\phi_m, s^{A_m}]$. We claim that $c(\phi, f)$ is dense in $[\phi_m, s^{A_m}]$. This is because $f \in F_{m,n}(\phi, s^A)$ implies $c(\phi, f) \cap [A_m^n, s^{A_m^n}] \neq \emptyset$, and because sets of the form $[\phi_m^n, s^{A_m^n}]$ form a basis of $[\phi_m, s^{A_m}]$. \square

\square

We will frequently denote structures in \mathcal{G} by $[m] = \{1, 2, \dots, m\}$, $[-k, -1] \cup [1, k] = \{-k, \dots, -1, 1, \dots, k\}$, etc. From now on, whenever we write s^A, s^B, s^C, s^D , we always mean the antidiagonal of A, B, C, D , respectively.

In the rest of the section we prove Proposition 3.2. We illustrate our proof in Example 3.15. We start with a simple lemma. The proof is straightforward.

Lemma 3.6. *Let $A = [-k, -1] \cup [1, k]$ and $B = [-l, -1] \cup [1, l]$. Let $\phi: B \rightarrow A$ be an epimorphism. Then $\phi: (B, s^B) \rightarrow (A, s^A)$ is an epimorphism if and only if for every i , $\phi(i) = -\phi(-i)$. In particular, if $\phi: (B, s^B) \rightarrow (A, s^A)$ is an epimorphism, then $\phi(1) = 1$ and $\phi(-1) = -1$, or $\phi(1) = -1$ and $\phi(-1) = 1$.*

Therefore, it is enough to show the following proposition.

Proposition 3.7. *Let $A = [-k, -1] \cup [1, k]$, $B = [-l, -1] \cup [1, l]$, and $C = [-m, -1] \cup [1, m]$. Let $\phi_1: B \rightarrow A$ and $\phi_2: C \rightarrow A$ be epimorphisms such that for every i , $\phi_1(i) = -\phi_1(-i)$ and $\phi_2(i) = -\phi_2(-i)$. Then there are $D = [1, n]$ and epimorphisms $\psi_1: D \rightarrow B$ and $\psi_2: D \rightarrow C$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$, and moreover $\psi_1(1) \in \{-1, 1\}$ and $\psi_2(1) \in \{-1, 1\}$.*

Proof that Proposition 3.7 implies Proposition 3.2. Take $A = [-k, -1] \cup [1, k]$, $B = [-l, -1] \cup [1, l]$, and $C = [-m, -1] \cup [1, m]$. Take epimorphisms $\phi_1: (B, s^B) \rightarrow (A, s^A)$ and $\phi_2: (C, s^C) \rightarrow (A, s^A)$. From Lemma 3.6, for every i , $\phi_1(i) = -\phi_1(-i)$ and $\phi_2(i) = -\phi_2(-i)$. From the conclusion of Proposition 3.7 get $D = [1, n]$, ψ_1, ψ_2 . Take $D' = [-n, -1] \cup [1, n]$ and let $s^{D'}$ be the antidiagonal of D' . Extend ψ_1 and ψ_2 to D' so that for every i , $\psi_1(i) = -\psi_1(-i)$ and $\psi_2(i) = -\psi_2(-i)$. Then $\psi_1: (D', s^{D'}) \rightarrow (B, s^B)$ and $\psi_2: (D', s^{D'}) \rightarrow (C, s^C)$ are epimorphisms and $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$. \square

To show Proposition 3.7, we need the Steinhaus' chessboard theorem. The Steinhaus' chessboard theorem was first used by Solecki to show the amalgamation property of the family of finite reflexive linear graphs (see Remark 3.11 for the sketch of his proof). We use the Steinhaus' chessboard theorem as one of the ingredients of the proof of Proposition 3.2.

For m, n positive define a *chessboard* to be $\mathbb{C} = [m] \times [n]$. The *boundary* of the chessboard \mathbb{C} , denoted by $\text{Bd}(\mathbb{C})$, is defined to be the set $(\{1, m\} \times [n]) \cup ([m] \times \{1, n\})$. For $(a_1, b_1), (a_2, b_2) \in \mathbb{C}$, we say that they are *8-adjacent* if they are different and $|a_1 - a_2| \leq 1$, and $|b_1 - b_2| \leq 1$; they are *4-adjacent* if they are different and either $|a_1 - a_2| \leq 1$ and $b_1 = b_2$, or $a_1 = a_2$ and $|b_1 - b_2| \leq 1$. A sequence x_1, x_2, \dots, x_l is a *4-path* (an *8-path*) from $A, B \subseteq \mathbb{C}$ if $x_1 \in A$, $x_l \in B$, and for every i , x_i and x_{i+1} are 4-adjacent (8-adjacent). For $x, y \in \text{Bd}(\mathbb{C})$, $x \neq y$, there are exactly two 4-paths from x to y such that every element in the path is in the boundary: clockwise and counter-clockwise. If $x = x_1, x_2, \dots, x_l = y$ is the clockwise path from x to y , we let $\vec{xy} = \{x_1, x_2, \dots, x_l\}$. For $x \in \text{Bd}(\mathbb{C})$, we let $\vec{xx} = \{x\}$. We say that $w, x, y, z \in \text{Bd}(\mathbb{C})$ is an *oriented quadruple* if $y, z \notin \vec{wx}$ and $z \notin \vec{xy}$. A *coloring* is any function $f: \mathbb{C} \rightarrow \{\text{black}, \text{white}\}$.

The theorem below is due to Hugo Steinhaus, for the proof we refer the reader to [8]. We use the chessboard theorem to obtain various amalgamation results (Lemma 3.10).

Theorem 3.8 (Steinhaus' chessboard theorem). *Let \mathbb{C} be a chessboard. Let $w, x, y, z \in Bd(\mathbb{C})$ be an oriented quadruple. Then for every coloring $f: \mathbb{C} \rightarrow \{\text{black}, \text{white}\}$ the existence of an 8-path from \overrightarrow{wx} to \overrightarrow{yz} is equivalent to the non-existence of a 4-path from \overrightarrow{xy} to \overrightarrow{zw} .*

Proposition 3.9. Let $A = [-k, -1] \cup [1, k]$, $B = [-l, -1] \cup [1, l]$, and $C = [-m, -1] \cup [1, m]$. Let $\phi_1: B \rightarrow A$ and $\phi_2: C \rightarrow A$ be epimorphisms such that for every i , $\phi_1(i) = -\phi_1(-i)$ and $\phi_2(i) = -\phi_2(-i)$. We let $(i, j) \in B \times C$ to be black if and only if $\phi_1(i) = \phi_2(j)$. Then there are: a black 8-path from $\{-1, 1\} \times \{-1, 1\}$ to $\{-l\} \times [-m, m]$, a black 8-path from $\{-1, 1\} \times \{-1, 1\}$ to $\{l\} \times [-m, m]$, a black 8-path from $\{-1, 1\} \times \{-1, 1\}$ to $[-l, l] \times \{-m\}$, and a black 8-path from $\{-1, 1\} \times \{-1, 1\}$ to $[-l, l] \times \{m\}$.

Proof that Proposition 3.9 implies Proposition 3.7. Let $A = [-k, -1] \cup [1, k]$, $B = [-l, -1] \cup [1, l]$, and $C = [-m, -1] \cup [1, m]$. Let $\phi_1: B \rightarrow A$ and $\phi_2: C \rightarrow A$ be epimorphisms such that for every i , $\phi_1(i) = -\phi_1(-i)$ and $\phi_2(i) = -\phi_2(-i)$. Observe that either $\{-1\} \times \{-1\}$ and $\{1\} \times \{1\}$ are black and $\{-1\} \times \{1\}$ and $\{1\} \times \{-1\}$ are white, or $\{-1\} \times \{-1\}$ and $\{1\} \times \{1\}$ are white and $\{-1\} \times \{1\}$ and $\{1\} \times \{-1\}$ are black. Without loss of generality, the former holds. Let w_1, w_2, \dots, w_p be a black 8-path from $\{1\} \times \{1\}$ to $\{-l\} \times [-m, m]$ (clearly, there is a black 8-path from $\{1\} \times \{1\}$ to $\{-l\} \times [-m, m]$ if and only if there is a black 8-path from $\{-1\} \times \{-1\}$ to $\{-l\} \times [-m, m]$). Let x_1, x_2, \dots, x_q be a black 8-path from $\{1\} \times \{1\}$ to $\{l\} \times [-m, m]$, let y_1, y_2, \dots, y_r be a black 8-path from $\{1\} \times \{1\}$ to $[-l, l] \times \{-m\}$, and let z_1, z_2, \dots, z_s be a black 8-path from $\{1\} \times \{1\}$ to $[-l, l] \times \{m\}$. Let

$$D = \{w_1, w_2, \dots, w_p, w_p, \dots, w_2, w_1, x_1, x_2, \dots, x_q, x_q, \dots, x_2, x_1, y_1, y_2, \dots, y_r, y_r, \dots, y_2, y_1, z_1, z_2, \dots, z_s, z_s, \dots, z_2, z_1\}.$$

For $t \in D$, if $t = (a, b)$, we let $\psi_1(t) = a$ and $\psi_2(t) = b$. This works. \square

In the rest of the section we prove Proposition 3.9. Let $[r]$, $[s]$, and $[t]$ be given. Let $\alpha: [s] \rightarrow [r]$ and $\beta: [t] \rightarrow [r]$ be relation preserving maps (not necessarily onto). For the chessboard $\mathbb{C} = [s] \times [t]$, we let (i, j) to be black if and only if $\alpha(i) = \beta(j)$. In the lemma below we collect amalgamation results we will use later.

Lemma 3.10. a) *If $\alpha(1) = 1$, $\beta(1) = \beta(t) = 1$, and $\text{rng}(\beta) \subseteq \text{rng}(\alpha)$, then there is a black 8-path from $(1, 1)$ to $(1, t)$.*

b) *If $\alpha(1) = 1$, $\beta(1) = 1$, and $\text{rng}(\beta) \subseteq \text{rng}(\alpha)$, then there is a black 8-path from $(1, 1)$ to $[s] \times \{t\}$.*

We will write a careful proof of part a). A proof of b) is very similar, and is left to the reader.

Proof. We show that there is a black 8-path from $(1, 1)$ to $(1, t)$. For this, via the Steinhaus' chessboard theorem, it is enough to show that there is no white 4-path from $\overrightarrow{(1, 1)(1, t)}$ to $\overrightarrow{(1, t)(1, 1)}$. Take a 4-path $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ from $\overrightarrow{(1, 1)(1, t)}$ to

$\overrightarrow{(1,t)(1,1)}$. We show that for some i , (a_i, b_i) is black, that is, $\alpha(a_i) = \beta(b_i)$. Define $h: \{1, 2, \dots, n\} \rightarrow \mathbb{Z}$, where \mathbb{Z} is the set of integers, by $h(i) = \alpha(i) - \beta(i)$. The function h has an important ‘continuity’ property: for every i , $|h(i+1) - h(i)| \leq 1$. We want to find i such that $h(i) = 0$. We will consider three cases: $(a_n, b_n) \in [s] \times \{1\}$, $(a_n, b_n) \in [s] \times \{t\}$, and $(a_n, b_n) \in \{s\} \times [t]$.

First, let $(a_n, b_n) \in [s] \times \{1\}$. Since $\alpha(a_1) = 1$, we have $h(1) \leq 0$, and since $\beta(b_n) = 1$, we have $h(n) \geq 0$. Therefore, by the ‘continuity’ property, for some i , $h(i) = 0$. In the case when $(a_n, b_n) \in [s] \times \{t\}$, for the same reason, there is i such that $h(i) = 0$. Suppose now that $(a_n, b_n) \in \{s\} \times [t]$. Let $[r_0] = \text{rng}(\alpha)$. Let $x, y \in [s]$ be such that $\alpha(x) = 1$ and $\alpha(y) = r_0$. Take (a_{i_0}, b_{i_0}) such that $a_{i_0} = x$ and take (a_{j_0}, b_{j_0}) such that $a_{j_0} = y$. Since $\text{rng}(\beta) \subseteq \text{rng}(\alpha)$, we have $\beta(b_{j_0}) \leq r_0$, and therefore $h(j_0) \geq 0$. Since also $h(i_0) \leq 0$, for some i we have $h(i) = 0$.

□

Remark 3.11. The Steinhaus’ chessboard theorem was used by Solecki to prove the AP of the family \mathcal{G} (unpublished). His proof is much simpler than the one presented in [3]. We give here a sketch of Solecki’s proof (with his permission). Let $A = [k], B = [l], C = [m] \in \mathcal{G}$ and epimorphisms $\alpha: B \rightarrow A$ and $\beta: B \rightarrow A$ be given. We want to find $D \in \mathcal{G}$, $\gamma: D \rightarrow B$, and $\delta: D \rightarrow C$ such that $\alpha \circ \gamma = \beta \circ \delta$. Consider the chessboard $[l] \times [m]$. Let (i, j) be black if and only if $\alpha(i) = \beta(j)$. An argument similar to the one we used in the proof of Lemma 3.10 shows that there is no white 4-path from $\{1\} \times [m]$ to $\{l\} \times [m]$, and there is no white 4-path from $[l] \times \{1\}$ to $[l] \times \{m\}$. Therefore, by the Steinhaus’ chessboard theorem, there are a black 8-path from $[l] \times \{1\}$ to $[l] \times \{m\}$ and a black 8-path from $\{1\} \times [m]$ to $\{l\} \times [m]$. We can combine these two 8-paths into one 8-path x_1, x_2, \dots, x_n with the following property: for every $a \in [l]$ there is $b \in [m]$ such that for some i , $(a, b) = x_i$, and for every $b \in [m]$ there is $a \in [l]$ such that for some i , $(a, b) = x_i$. Let $D = [n]$. For $i \in [n]$, if $x_i = (a, b)$, we let $\gamma(i) = a$ and $\delta(i) = b$. This works.

Let A, B, C and ϕ_1, ϕ_2 be as in the hypotheses of Proposition 3.9. We pick points in $B = [-l, -1] \cup [1, l]$:

$$s_{-p} < s'_{-p+1} < \dots < s'_{-2} < s_{-2} < s'_{-1} < s_{-1} < s'_0 < s_0 < s'_1 < s_1 < s'_2 < s_2 < \dots < s'_p,$$

for some p , so that $s_0 = 1$, $s'_0 = -1$, $s'_p = l$, $s_{-p} = -l$, for every $-p < i < p$ we have $s_i = s'_i + 1$, for every $-p \leq i < p$ the epimorphism ϕ_1 assumes only positive values or assumes only negative values in the interval $[s_i, s'_{i+1}]$, and for every $-p < i < p$, $\phi_1(s'_i)$ and $\phi_1(s_i)$ have opposite signs.

Notice that since $\phi_1(j) = -\phi_1(-j)$, $j \in B$, we have for every $-p \leq i < p$, $s_i = -s'_{-i}$. Notice also that if $\phi_1(s_0) > 0$ then for any even number $-p \leq i < p$ we have $\phi_1(s_i) = \phi_1(s'_{i+1}) = 1$ and for any odd number $-p \leq i < p$ we have $\phi_1(s_i) = \phi_1(s'_{i+1}) = -1$. On the other hand, if $\phi_1(s_0) < 0$ then for any even number $-p \leq i < p$ we have $\phi_1(s_i) = \phi_1(s'_{i+1}) = -1$ and for any odd number $-p \leq i < p$ we have $\phi_1(s_i) = \phi_1(s'_{i+1}) = 1$.

Similarly, we pick points in $C = [-m, -1] \cup [1, m]$:

$$t_{-q} < t'_{-q+1} < \dots < t'_{-2} < t_{-2} < t'_{-1} < t_{-1} < t'_0 < t_0 < t'_1 < t_1 < t'_2 < t_2 < \dots < t'_q,$$

for some q , so that $t_0 = 1$, $t'_0 = -1$, $t'_q = m$, $t_{-q} = -m$, for every $-q < i < q$ we have $t_i = t'_i + 1$, for every $-q \leq i < q$ the epimorphism ϕ_2 assumes only positive values or assumes only negative values in the interval $[t_i, t'_{i+1}]$, and for every $-q < i < q$, $\phi_2(t'_i)$ and $\phi_2(t_i)$ have opposite signs.

Notice that since $\phi_2(j) = -\phi_2(-j)$, $j \in C$, we have for every $-q \leq i < q$, $t_i = -t'_{-i}$. Notice also that if $\phi_2(t_0) > 0$ then for any even number $-q \leq i < q$ we have $\phi_2(t_i) = \phi_2(t'_{i+1}) = 1$ and for any odd number $-q \leq i < q$ we have $\phi_2(t_i) = \phi_2(t'_{i+1}) = -1$. On the other hand, if $\phi_2(t_0) < 0$ then for any even number $-q \leq i < q$ we have $\phi_2(t_i) = \phi_2(t'_{i+1}) = -1$ and for any odd number $-q \leq i < q$ we have $\phi_2(t_i) = \phi_2(t'_{i+1}) = 1$.

We define a graph \mathcal{G}_0 . Let the set of vertices in \mathcal{G}_0 be equal to the set $V = [-p, p] \times [-q, q]$. For $(a, b), (c, d) \in V$, if they are not 4-adjacent, they will not be connected by an edge. For every $-p \leq i < p$ and $-q \leq j < q$ such that ϕ_1 has the same sign on $[s_i, s'_{i+1}]$ as ϕ_2 has on $[t_j, t'_{j+1}]$, if $\text{rng}(\phi_1 \upharpoonright [s_i, s'_{i+1}]) \subseteq \text{rng}(\phi_2 \upharpoonright [t_j, t'_{j+1}])$, we put an edge between (i, j) and $(i+1, j)$ and we put an edge between $(i, j+1)$ and $(i+1, j+1)$. If $\text{rng}(\phi_2 \upharpoonright [t_j, t'_{j+1}]) \subseteq \text{rng}(\phi_1 \upharpoonright [s_i, s'_{i+1}])$, we put an edge between (i, j) and $(i, j+1)$ and we put an edge between $(i+1, j)$ and $(i+1, j+1)$. We have just defined \mathcal{G}_0 .

Let x_1, x_2, \dots, x_n be a path in the chessboard $[-p, p] \times [-q, q]$. We say that it is *interior* if $x_1, \dots, x_{n-1} \notin \text{Bd}([-p, p] \times [-q, q])$.

Lemma 3.12. (1) *The existence of an interior path in \mathcal{G}_0 (in the graph-theoretic sense) from $(0, 0)$ to $[-p, p] \times \{-q\}$ implies the existence of a black 8-path in $B \times C$ from $\{-1, 1\} \times \{-1, 1\}$ to $[-l, l] \times \{-m\}$.*
(2) *The existence of an interior path in \mathcal{G}_0 from $(0, 0)$ to $[-p, p] \times \{q\}$ implies the existence of a black 8-path in $B \times C$ from $\{-1, 1\} \times \{-1, 1\}$ to $[-l, l] \times \{m\}$.*
(3) *The existence of an interior path in the graph \mathcal{G}_0 from $(0, 0)$ to $\{-p\} \times [-q, q]$, implies the existence of a black 8-path in $B \times C$ from $\{-1, 1\} \times \{-1, 1\}$ to $\{-l\} \times [-m, m]$.*
(4) *The existence of an interior path in \mathcal{G}_0 from $(0, 0)$ to $\{p\} \times [-q, q]$ implies the existence of a black 8-path in $B \times C$ from $\{-1, 1\} \times \{-1, 1\}$ to $\{l\} \times [-m, m]$.*

Proof. This follows from Lemma 3.10 and from the definition of \mathcal{G}_0 . \square

To finish the proof of Proposition 3.9, we have to show that there are paths in \mathcal{G}_0 , as in the hypotheses of Lemma 3.12.

Define \mathcal{G}_1 to be a subgraph of \mathcal{G}_0 such that for every $-p \leq i < p$ and $-q \leq j < q$ such that ϕ_1 has the same sign on $[s_i, s'_{i+1}]$ as ϕ_2 has on $[t_j, t'_{j+1}]$ if $\text{rng}(\phi_2 \upharpoonright [t_j, t'_{j+1}]) = \text{rng}(\phi_1 \upharpoonright [s_i, s'_{i+1}])$, we delete the edge between (i, j) and $(i+1, j)$, we delete the edge between $(i, j+1)$ and $(i+1, j+1)$, keep the edge between (i, j) and $(i, j+1)$, and we keep the edge between $(i+1, j)$ and $(i+1, j+1)$.

Define \mathcal{G}_2 to be a subgraph of \mathcal{G}_0 such that for every $-p \leq i < p$ and $-q \leq j < q$ such that ϕ_1 has the same sign on $[s_i, s'_{i+1}]$ as ϕ_2 has on $[t_j, t'_{j+1}]$ if $\text{rng}(\phi_2 \upharpoonright [t_j, t'_{j+1}]) = \text{rng}(\phi_1 \upharpoonright [s_i, s'_{i+1}])$, we keep the edge between (i, j) and $(i+1, j)$, we keep the edge between $(i, j+1)$ and $(i+1, j+1)$, delete the edge between (i, j) and $(i, j+1)$, and we delete the edge between $(i+1, j)$ and $(i+1, j+1)$.

Remark 3.13. In the graph \mathcal{G}_0 , (a, b) and (c, d) are connected by an edge if and only if $(-a, -b)$ and $(-c, -d)$ are connected by an edge. The same conclusion holds for graphs \mathcal{G}_1 and \mathcal{G}_2 .

The existence of the required paths in \mathcal{G}_0 will follow from the lemma below.

Lemma 3.14. (1) *In the graph \mathcal{G}_1 there is an interior path from $(0, 0)$ to $[-p, p] \times \{-q\}$ and there is an interior path from $(0, 0)$ to $[-p, p] \times \{q\}$.*
(2) *In the graph \mathcal{G}_2 there is an interior path from $(0, 0)$ to $\{-p\} \times [-q, q]$ and there is an interior path from $(0, 0)$ to $\{p\} \times [-q, q]$.*

Proof. We show (1). The proof of (2) is similar.

Claim 1. For every $-p < i < p$, $-q < j < q$ there are exactly two edges that end in (i, j) .

Proof. Fix (i, j) . Either ϕ_1 has the same sign on $[s_i, s'_{i+1}]$ as ϕ_2 has on $[t_j, t'_{j+1}]$ and ϕ_1 has the same sign on $[s_{i-1}, s'_i]$ as ϕ_2 has on $[t_{j-1}, t'_j]$, or ϕ_1 has the same sign on $[s_i, s'_{i+1}]$ as ϕ_2 has on $[t_{j-1}, t'_j]$ and ϕ_1 has the same sign on $[s_{i-1}, s'_i]$ as ϕ_2 has on $[t_j, t'_{j+1}]$ (exactly one of these two possibilities hold). Say, the former holds. Since ϕ_1 has the same sign on $[s_i, s'_{i+1}]$ as ϕ_2 has on $[t_j, t'_{j+1}]$, by the definition of \mathcal{G}_1 , there is an edge between (i, j) and $(i+1, j)$, or there is an edge between (i, j) and $(i, j+1)$ (exactly one of these two possibilities hold). Since ϕ_1 has the same sign on $[s_{i-1}, s'_i]$ as ϕ_2 has on $[t_{j-1}, t'_j]$, there is an edge between (i, j) and $(i-1, j)$, or there is an edge between (i, j) and $(i, j-1)$ (exactly one of these two possibilities hold). This finishes the proof of the claim. \square

Claim 2. There are no loops passing through $(0, 0)$ in the graph \mathcal{G}_1 .

Proof. Suppose towards a contradiction that $(0, 0) = (a_0, b_0), (a_1, b_1), \dots, (a_n, b_n) = (0, 0)$, where $(a_0, b_0), (a_1, b_1), \dots, (a_{n-1}, b_{n-1})$ are pairwise different, is a loop. By Claim 1 and Remark 3.13, $(a_{n-1}, b_{n-1}) = (-a_1, -b_1)$, $(a_{n-2}, b_{n-2}) = (-a_2, -b_2), \dots$. Hence, if n is even, we have $(a_{\frac{n}{2}}, b_{\frac{n}{2}}) = (-a_{\frac{n}{2}}, -b_{\frac{n}{2}})$. This implies $(a_{\frac{n}{2}}, b_{\frac{n}{2}}) = (0, 0)$ and contradicts the assumption that $(a_{\frac{n}{2}}, b_{\frac{n}{2}}) \neq (a_0, b_0)$. If n is odd, we get $(a_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}) = (-a_{\frac{n+1}{2}}, -b_{\frac{n+1}{2}})$. Since $(a_{\frac{n-1}{2}}, b_{\frac{n-1}{2}})$ and $(a_{\frac{n+1}{2}}, b_{\frac{n+1}{2}})$ are 4-adjacent, and since $(a_{\frac{n-1}{2}}, b_{\frac{n-1}{2}}) \neq (0, 0)$, we again arrive at a contradiction. \square

Notice that Claim 1 and Claim 2 already imply that there is a path from $(0, 0)$ to the boundary of $[-p, p] \times [-q, q]$.

Let $-p \leq i_0 < p$ be such that $\text{rng}(\phi_1 \upharpoonright [s_{i_0}, s'_{i_0+1}]) = [1, k]$ (recall that $A = [-k, -1] \cup [1, k]$). Then $\text{rng}(\phi_1 \upharpoonright [s_{-(i_0+1)}, s'_{-i_0}]) = [-k, -1]$. Therefore, for every $-q \leq j < q$, if ϕ_1 has the same sign on $[s_{i_0}, s'_{i_0+1}]$ as ϕ_2 has on $[t_j, t'_{j+1}]$, then $\text{rng}(\phi_2 \upharpoonright [t_j, t'_{j+1}]) \subseteq \text{rng}(\phi_1 \upharpoonright [s_{i_0}, s'_{i_0+1}])$, and if ϕ_1 has the same sign on $[s_{-(i_0+1)}, s'_{-i_0}]$ as ϕ_2 has on $[t_j, t'_{j+1}]$, then $\text{rng}(\phi_2 \upharpoonright [t_j, t'_{j+1}]) \subseteq \text{rng}(\phi_1 \upharpoonright [s_{-(i_0+1)}, s'_{-i_0}])$. Therefore, for every j , there is no edge between (i_0, j) and $(i_0 + 1, j)$ and there is no edge between $(-i_0 + 1, j)$ and $(-i_0, j)$.

This implies that the path from $(0, 0)$ to the boundary of $[-p, p] \times [-q, q]$ is in fact from $(0, 0)$ to $[-i_0, i_0] \times \{-q, q\}$. Since $\phi_1(i) = -\phi_1(-i)$ and $\phi_2(i) = -\phi_2(-i)$, the existence of a path from $(0, 0)$ to $[-p, p] \times \{-q\}$ is equivalent to the existence of a path from $(0, 0)$ to $[-p, p] \times \{q\}$. Therefore, there exist paths from $(0, 0)$ to $[-p, p] \times \{-q\}$ and from $(0, 0)$ to $[-p, p] \times \{q\}$. Without loss of generality, only the last element of each path is in $\text{Bd}([-p, p] \times [-q, q])$. \square

Example 3.15. We illustrate the proof of Theorem 3.1 on an example. Let $A = [-3, -1] \cup [1, 3]$, $B = [-8, -1] \cup [1, 8]$, $C = [-9, -1] \cup [1, 9]$. Let $\phi_1: B \rightarrow A$ be given by $\phi_1(1) = 1$, $\phi_1(2) = 2$, $\phi_1(3) = 1$, $\phi_1(4) = -1$, $\phi_1(5) = 1$, $\phi_1(6) = 1$, $\phi_1(7) = 2$, $\phi_1(8) = 3$, and $\phi_1(-i) = \phi_1(i)$ for $i \in [1, 8]$. Let $\phi_2: C \rightarrow A$ be given by $\phi_2(1) = -1$, $\phi_2(2) = -2$, $\phi_2(3) = -1$, $\phi_2(4) = -2$, $\phi_2(5) = -3$, $\phi_2(6) = -2$, $\phi_2(7) = -1$, $\phi_2(8) = 1$, $\phi_2(9) = 2$, and $\phi_2(-i) = \phi_2(i)$ for $i \in [1, 9]$.

We consider the chessboard $B \times C$, where (i, j) is black if and only if $\phi_1(i) = \phi_2(j)$ (Figure 1).

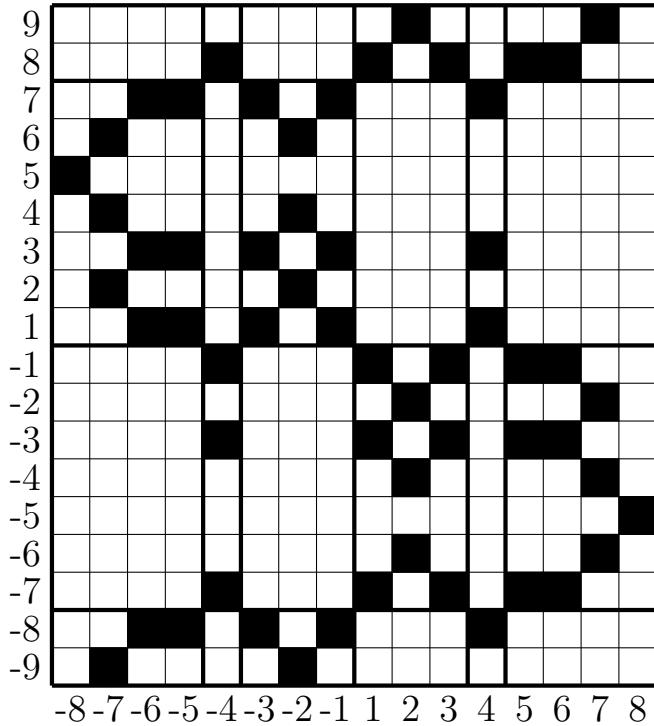
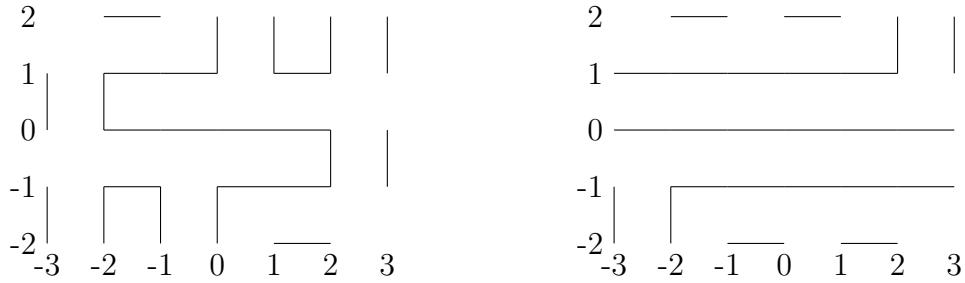
Therefore, $s_{-3} = -8$, $s'_{-2} = -5$, $s_{-2} = -4$, $s'_{-1} = -4$, $s_{-1} = -3$, $s'_0 = -1$, $s_0 = 1$, $s'_1 = 3$, $s_1 = 4$, $s'_2 = 4$, $s_2 = 5$, $s'_3 = 8$, and $t_{-2} = -9$, $t'_{-1} = -8$, $t_{-1} = -7$, $t'_0 = -1$, $t_0 = 1$, $t'_1 = 7$, $t_1 = 8$, $t'_2 = 9$.

Moreover, $\text{rng}(\phi_1 \upharpoonright [s_{-3}, s'_{-2}]) = [-3, -1]$, $\text{rng}(\phi_1 \upharpoonright [s_{-2}, s'_{-1}]) = [1]$, $\text{rng}(\phi_1 \upharpoonright [s_{-1}, s'_0]) = [-2, -1]$, $\text{rng}(\phi_1 \upharpoonright [s_0, s'_1]) = [1, 2]$, $\text{rng}(\phi_1 \upharpoonright [s_1, s'_2]) = [-1]$, $\text{rng}(\phi_1 \upharpoonright [s_2, s'_3]) = [1, 3]$, and $\text{rng}(\phi_2 \upharpoonright [t_{-2}, t'_{-1}]) = [-2, -1]$, $\text{rng}(\phi_2 \upharpoonright [t_{-1}, t'_0]) = [1, 3]$, $\text{rng}(\phi_2 \upharpoonright [t_0, t'_1]) = [-3, -1]$, $\text{rng}(\phi_2 \upharpoonright [t_1, t'_2]) = [1, 2]$.

Hence, graphs \mathcal{G}_1 and \mathcal{G}_2 are as in Figure 2.

APPENDIX A.

The purpose of this appendix is to present a criterium for the automorphism group of a projective Fraïssé limit to have a dense conjugacy class, and to present a criterium for the automorphism group of a projective Fraïssé limit to have a comeager conjugacy class. These criteria and their proofs are analogs of Theorems 2.1 and 3.4 given by Kechris and Rosendal in the context of (injective) Fraïssé limits. However, we point out that we will work with surjective relations rather than with partial functions, our criteria are analogs but not dualizations of the corresponding criteria in [4]. It seems that working with surjective relations rather than with partial functions makes calculations simpler in

FIGURE 1. The chessboard $B \times C$ FIGURE 2. Graphs \mathcal{G}_1 (left) and \mathcal{G}_2 (right)

the context of projective Fraïssé limits. We hope that many new interesting projective Fraïssé limits will be discovered, and these criteria will be useful for them.

Let \mathcal{G} be a countable projective Fraïssé family in a language L_0 . Let \mathbb{P} be the projective Fraïssé limit of \mathcal{G} . Define

$$\begin{aligned} \mathcal{F} = & \{(A, s^A) : A \in \mathcal{G} \text{ and } \exists \phi: \mathbb{P} \rightarrow A \exists f \in \text{Aut}(\mathbb{P}) \text{ such that} \\ & \phi: (\mathbb{P}, f) \rightarrow (A, s^A) \text{ is an epimorphism}\}. \end{aligned}$$

Theorem A.1. *Let \mathcal{G} , \mathbb{P} , and \mathcal{F} be as above. Then \mathcal{F} has the JPP if and only if $\text{Aut}(\mathbb{P})$ has a dense conjugacy class.*

Proof. The proof that the JPP implies $\text{Aut}(\mathbb{P})$ has a dense conjugacy class is the same as in the special case (Proposition 2.6).

We show the converse. Take $(A, s^A), (B, s^B) \in \mathcal{F}$. We find $(C, s^C) \in \mathcal{F}$ such that there are epimorphisms from (C, s^C) onto (A, s^A) and onto (B, s^B) . Take any epimorphism $\phi: \mathbb{P} \rightarrow A$. Take $f \in [\phi, s^A]$ that has a dense conjugacy class. Take any epimorphism $\psi: \mathbb{P} \rightarrow B$. Let $g \in \text{Aut}(\mathbb{P})$ be such that $gfg^{-1} \in [\psi, s^B]$. Let C be a partition of \mathbb{P} that refines partitions $\phi^{-1}(A)$ and $g(\psi^{-1}(B))$, and moreover $r^\mathbb{P}$ restricted to C is in \mathcal{G} . (To achieve this last requirement on C , we use (L2).) Let $\hat{\phi}$ be the natural projection from \mathbb{P} to C . We let $s^C(c, d)$ if and only if there are $x, y \in \mathbb{P}$ such that $\hat{\phi}(x) = c$, $\hat{\phi}(y) = d$, and $f(x) = y$. Clearly, the natural projection $\bar{\phi}$ from (C, s^C) onto (A, s^A) is an epimorphism. Let $\hat{\psi} = \hat{\phi} \circ g^{-1}$. Let $\bar{\psi}$ be the natural projection from $(g^{-1}(C), g^{-1}(s^C))$ to (B, s^B) . Since there are $x, y \in \mathbb{P}$ such that $\hat{\phi}(x) = c$, $\hat{\phi}(y) = d$, and $f(x) = y$ if and only if there are $x, y \in \mathbb{P}$ such that $\hat{\psi}(x) = g^{-1}(c)$, $\hat{\psi}(y) = g^{-1}(d)$, and $gfg^{-1}(x) = y$, this projection is an epimorphism. \square

We say that a family \mathcal{F} of topological L -structures has *the weak amalgamation property*, or *the WAP*, if for every $A \in \mathcal{F}$ there is $B \in \mathcal{F}$ and an epimorphism $\phi: B \rightarrow A$ such that for any $C_1, C_2 \in \mathcal{F}$ and any epimorphisms $\phi_1: C_1 \rightarrow B$ and $\phi_2: C_2 \rightarrow B$, there exist $D \in \mathcal{F}$, $\phi_3: D \rightarrow C_1$, and $\phi_4: D \rightarrow C_2$ such that $\phi \circ \phi_1 \circ \phi_3 = \phi \circ \phi_2 \circ \phi_4$.

Theorem A.2. *Let \mathcal{G} , \mathbb{P} , and \mathcal{F} be as above. Then \mathcal{F} has the JPP and the WAP if and only if $\text{Aut}(\mathbb{P})$ has a comeager conjugacy class.*

Proof. The proof that the JPP and the CAP imply $\text{Aut}(\mathbb{P})$ has a comeager conjugacy class is the same as in the special case (Proposition 3.3). To show that the JPP and the WAP imply $\text{Aut}(\mathbb{P})$ has a comeager conjugacy class we have to make small modifications. We take the following definition of $E(\phi, s^A)$:

$$E(\phi, s^A) = \{f \in \text{Aut}(\mathbb{P}): \text{ if } f \in [\phi, s^A] \text{ then for some } ((B, s^B), \psi, \bar{\psi}), \text{ an extension of } ((A, s^A), \phi), \text{ we have for some } m, f \in [\psi_m, s^{B_m}]\},$$

then the proof goes through.

We now show the converse. Let $(M, s^M) \in \mathcal{F}$. Let $\phi: \mathbb{P} \rightarrow M$ be an epimorphism. Take $f \in [M, s^M]$ that has a comeager conjugacy class. By Proposition 3.5, there is an open neighborhood U of f such that $c(\phi, f) = \{gfg^{-1}: g \in [\phi, \text{id}_M]\}$ is dense in U . Therefore, there is $((N, s^N), \psi, \bar{\psi})$, an extension of $(M, s^M), \phi$, (the definition of being an extension is given just before the proof of Proposition 3.3) such that $c(\phi, f)$ is dense in $[\psi, s^N]$. We show that (N, s^N) works for (M, s^M) . For this, take $(A, s^A), (B, s^B) \in \mathcal{F}$ and

epimorphisms $\alpha: (A, s^A) \rightarrow (N, s^N)$ and $\beta: (B, s^B) \rightarrow (N, s^N)$. We find $(C, s^C) \in \mathcal{F}$ and epimorphisms $\gamma: (C, s^C) \rightarrow (A, s^A)$ and $\delta: (C, s^C) \rightarrow (B, s^B)$ such that $\bar{\psi} \circ \alpha \circ \gamma = \bar{\psi} \circ \beta \circ \delta$.

Take any epimorphism $\phi_1: \mathbb{P} \rightarrow A$. Take $f' \in [\phi_1, s^A]$, $f' = g_0 f g_0^{-1}$ for some $g_0 \in [\phi, \text{id}_M]$. Take any epimorphism $\phi_2: \mathbb{P} \rightarrow B$. Let $g \in [\phi, \text{id}_M]$ be such that $gf'g^{-1} \in [\phi_2, s^B]$. Let C be a partition of \mathbb{P} that refines partitions $\phi_1^{-1}(A)$ and $g(\phi_2^{-1}(B))$, and moreover $r^{\mathbb{P}}$ restricted to C is in \mathcal{G} . (To achieve this last requirement on C , we use (L2).) Let $\hat{\phi}_1$ be the natural projection from \mathbb{P} to C . We let $s^C(c, d)$ if and only if there are $x, y \in \mathbb{P}$ such that $\hat{\phi}_1(x) = c$, $\hat{\phi}_1(y) = d$, and $f'(x) = y$. Clearly, the natural projection $\bar{\phi}_1$ from (C, s^C) onto (A, s^A) is an epimorphism. Let $\hat{\phi}_2 = \hat{\phi}_1 \circ g^{-1}$. Let $\bar{\phi}_2$ be the natural projection from $(g^{-1}(C), g^{-1}(s^C))$ to (B, s^B) . Since there are $x, y \in \mathbb{P}$ such that $\hat{\phi}_2(x) = g^{-1}(c)$, $\hat{\phi}_2(y) = g^{-1}(d)$, and $gf'g^{-1}(x) = y$, this projection is an epimorphism. Let $\gamma = \bar{\phi}_1$ and $\delta = \bar{\phi}_2 \circ g$. Then $\bar{\psi} \circ \alpha \circ \gamma = \bar{\psi} \circ \beta \circ \delta$.

□

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